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On Almost Chebyshev Subspaces

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A closed subspace F in a Banach space X is called almost Chebyshev if the set of $x \in X$ which fail to have unique best approximation in F is contained in a first category subset. We prove, among other results, that if X is a separable Banach space which is either locally uniformly convex or has the Radon-Nikodym property, then "almost all" closed subspaces are almost Chebyshev.

1. INTRODUCTION

Let K be a nonempty subset of a Banach space X. For each $x \in X$, we say that $y \in K$ is a *best approximation* to x from K if

$$\| x - y \| = \inf\{\| x - z \| : z \in K\}.$$

The set K is said to have *property* U_x if best approximation in K with respect to x is unique. K is called *Chebyshev* if it has property U_x for each $x \in X$. When K is a closed subspace, we call it a *Chebyshev subspace*. It is known that if X is strictly convex, then any finite-dimensional subspace is Chebyshev. There exist separable nonstrictly convex spaces which do not have any finite-dimensional or finite-codimensional Chebyshev subspace (e.g., $L^1[0, 1]$ [7, 9]) and there are some without any infinite-dimensional Chebyshev subspace (e.g., c_0 [3]). In connection with this, there arises the question of whether a Banach space contains subspaces that are "close" to Chebyshev subspaces.

In [10], Stečkin introduced the concept of "almost Chebyshev." A set K is called *almost Chebyshev* if the set of x in X such that K fails to have property U_x is contained in a set of first category. He proved that if X is a uniformly convex Banach space, then every closed subset in X is almost Chebyshev. In [3], Garkavi showed that if X is separable, then for any reflexive subspace F in X, there exists an almost Chebyshev subspace (in fact, many) E in X which is B-isomorphic to F.

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In this paper, we will study the almost Chebyshev subspaces in certain classes of Banach spaces: those with the locally uniformly convex norms and those with the Radon–Nikodym property (RNP) [2, 5, 8]. The latter class contains all reflexive Banach spaces, dual separable Banach spaces, or more generally, dual Banach spaces which are weakly compact generated. Recently, Sundaresan [11] (cf. also [12]) showed that if $(S, \mathscr{B}, \mu)$ is a finite measure space and if X is a Banach space with the RNP, then $L_p(S, \mathscr{B}, \mu, X)$ also has the RNP for $1 < p < \infty$. Our main result is that if X is a separable Banach space which is locally uniformly convex or has the Radon–Nikodym property, then "almost all" closed subspaces are almost Chebyshev (Theorems 3.5, 3.7, Corollary 3.8).

In Section 2, we introduce some definitions and lemmas. We prove the main theorem in Section 3. Section 4 is for some remarks and open questions.

2. Definitions and Preliminaries

Throughout we will consider real Banach spaces: we use X^* to denote the dual of X. Suppose K is a convex subset in X, a point $x \in K$ is called an *exposed point* of K if there exists an $f \in X^*$ such that $f(x) > f(y)$ for all $y \in K$, $y \neq x$. It is called a *strongly exposed point* of K it is an exposed point and satisfies: for $\{x_n\} \subseteq K$, $f(x_n) \to f(x)$, then $x_n \to x$ in norm. The corresponding functionals to the strongly exposed points in K are called *strongly exposing functionals*. We use $K^{\perp}$ to denote the set of strongly exposing functionals of K.

A Banach space X is said to have the *Radon–Nikodym property* (RNP) if for any given σ-algebra $\mathscr{B}$ on a set Ω, any finite positive measure μ on $\mathscr{B}$, and any X-valued measure m on $\mathscr{B}$ of finite total variation absolutely continuous with respect to μ, there exists an X-valued Bochner measurable function f: $\Omega \to X$ such that $m(E) = \int_E f\, d\mu$ for $E \in \mathscr{B}$ (cf., e.g., [2]). One of the geometric characterizations of such spaces, which is relevant in here, is that [8]: *every bounded closed convex subset is the closed convex hull of its strongly exposed points*. In [5, 6], it is observed that

Proposition 2.1. *Let X be a Banach space with the RNP. Then for any bounded closed convex subset K in X, the set of strongly exposing functionals $K^{\perp}$ of K is a dense G_δ in X^*.*

A Banach space is called *locally uniformly convex* if for any x in X with $\|x\| = 1$ and $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\|x - y\| > \epsilon$ with $\|y\| = 1$, $\|x + y\| < 2(1 - \delta)$. It follows easily from the definition that each boundary point of the closed unit sphere S of a locally uniformly convex space is a strongly exposed point of S.

PROPOSITION 2.2. *Let X be a locally uniformly convex space, then S^{Δ} is a dense G_δ in X^*.*

Proof. The above remark shows that every support functional of S is also a strongly exposing functional. By the theorem of Bishop and Phelps on support functionals [1], the set S^{Δ} is dense in X^*. That S^{Δ} is a dense G_δ follows that for $n = 1, 2,...,$ the sets

$$G_n = \{f \in X^* : \operatorname{diam}\{x \in S : f(x) > \|f\| - a\} < 1/n \text{ for some } a > 0\}$$

are open and $S^{\Delta} = \bigcap_{n=1}^{\infty} G_n$.

DEFNIITION 2.3. A Banach space X is said to have property (P) if for each closed subspace F in X, the set of strongly exposing functionals of the closed unit sphere of F is a dense G_δ in F^*.

Note that the Radon–Nikodym property and locally uniform convexity are hereditary. Propositions 2.1 and 2.2 show that these two classes of Banach spaces have property (P).

To conclude this section, we will prove a topological lemma. A Hausdorff topological space X is called a *Baire space* if the intersection of any sequence of open dense subsets in X is again dense in X. It is easy to show that any complete metric space is a Baire space. Suppose X, Y are two sets and suppose G is a subset in $X \times Y$. For each $y \in Y$, we use G_y , the y-section of G, to denote the set $\{x \in X : (x, y) \in G\}$.

LEMMA 2.4. *Let X be a complete separable metric space and let Y be a Baire space. Suppose G is a dense G_δ subset in $X \times Y$; let*

$$A = \{y \in Y : G_y \text{ is a dense } G_\delta \text{ in } X\}.$$

Then A is a dense G_δ in Y.

Proof. We may assume that G is an open dense subset in $X \times Y$; the general case follows by taking countable intersection. Let $G = \bigcup_i (U_i \times V_i)$ where U_i and V_i are open subsets of X, Y, respectively. Let $\{x_n\}$ be a countable dense set in X and for each m, n, let $N(x_n , 1/m)$ be the neighborhood of x_n with radius $1/m$. Let

$$A_{mn} = \bigcup_i \{V_i : U_i \cap N(x_n , 1/m) \neq \varnothing\}.$$

We claim that A_{mn} is dense in Y. For otherwise, we can find an open subset W in Y such that $W \cap A_{mn} = \varnothing$. This implies $(N(x_n , 1/m) \times W) \cap G = \varnothing$, contradicting that G is a dense set in $X \times Y$. Note that each A_{mn} is open, hence $A = \bigcap_{m,n} A_{mn}$ is a dense G_δ in Y. It remains to show that for each $y \in A$, G_y is a dense G_δ in X. Indeed, for any m, n there exists $U_i \times V_i \subseteq G$

such that $y \in V_i$ and $x \in N(x_n, 1/m) \cap U_i$ ($\neq \varnothing$), i.e., G_y is dense in X. We complete the proof by observing that G_y is open in X (for G is assumed open).

3. Almost Chebyshev Subspaces

Let X be a Banach space, we use Θ to denote the family of closed subspaces in X. For $E, F \in \Theta$, define

$$\rho(E, F) = \max\{\sup_{\substack{\|y\| \leq 1 \\ y \in F}} \inf_{\substack{\|x\| \leq 1 \\ x \in E}} \|x - y\|, \sup_{\substack{\|x\| \leq 1 \\ x \in E}} \inf_{\substack{\|y\| \leq 1 \\ y \in F}} \|x - y\|\}.$$

It is proved in [4] that (Θ, ρ) is a complete metric space. For $E, F \in \Theta$, we say that E, F are *B-isomorphic* if there exists an isomorphism T from X onto X with $T(E) = F$ [3]. Let $\Theta(F)$ denote the family of closed subspaces in X which are B-isomorphic to F. In [3], Garkavi introduced another metric $\tilde{\rho}$ on $\Theta(F)$:

$$\tilde{\rho}(E, E') = \inf_{T} \left\{\sup_{x \neq 0} \left\| \frac{x}{\|x\|} - \frac{Tx}{\|Tx\|} \right\| + \log \max\{\|T\|, \|T^{-1}\|\}\right\}$$

where $E, E' \in \Theta(F)$ and the infimum is taken over all B-isomorphisms T from E onto E'. He also proved that $(\Theta(F), \tilde{\rho})$ is a complete metric space and that $\tilde{\rho}$ is stronger than ρ on $\Theta(F)$ (in fact, $\rho \leq \tilde{\rho}$); if F is of finite dimension or finite codimension in X, then ρ and $\tilde{\rho}$ are equivalent.

Lemma 3.1. *Let X be a Banach space, let E, F be hyperplanes in X defined by the functionals $f, g \in X^*$ ($\|f\| = \|g\| = 1$) as $E = f^{-1}(0)$, $F = g^{-1}(0)$. Then*

(i) *there exists an isomorphism $T : X \to X$ with $T(E) = F$ and*

$$\max\{\|T\|, \|T^{-1}\|\} \leq 1 + 4\|f - g\|;$$

(ii) *if $0 < \epsilon < \frac{1}{8}$ and $\|f - g\| < \epsilon$, then $\tilde{\rho}(E, F) < 20\epsilon$;*

(iii) *if X is a subspace of another Banach space X_1, consider E, F as subspaces in X_1; then assertions* (i), (ii) *still hold.*

Proof. Assertion (a) is proved in [3, Lemma II]; the isomorphism T: $X \to X$ with $T(E) = F$ can be chosen as

$$Tx = x + (f(x) - g(x))z, \qquad x \in X,$$

where z satisfies

$$f(z) = g(z) = 1 \qquad \text{and} \qquad \|z\| \leq 4$$

and T^{-1} is given by

$$T^{-1}x = x + (g(x) - f(x))z.$$

To prove (ii), we first estimate the quantity

$$\left\| \frac{x}{\| x \|} - \frac{Tx}{\| Tx \|} \right\|, \qquad x \in E.$$

Without loss of generality, assume $\| x \| = 1$. Hence

$$\begin{aligned} \left\| x - \frac{Tx}{\| Tx \|} \right\| &= \left\| x - \frac{x + (f(x) - g(x))\, z}{\| x + (f(x) - g(x))\, z \|} \right\| \\ &< \frac{1 - \| x + (f(x) - g(x))\, z \|}{1 - 4\epsilon} + \frac{| f(x) - g(x) | \cdot \| z \|}{1 - 4\epsilon} \\ &\leqslant \frac{4\epsilon}{1 - 4\epsilon} + \frac{4\epsilon}{1 - 4\epsilon} \\ &< 16\epsilon. \end{aligned}$$

By (i), we know that $\max\{\| T \|, \| T^{-1} \|\} \leqslant 1 + 4\epsilon$. Hence

$$\tilde{\rho}(E, F) \leqslant \sup_{x \neq 0} \left\{ \left\| \frac{x}{\| x \|} - \frac{Tx}{\| Tx \|} \right\| + \log \max\{\| T \|, \| T^{-1} \|\} \right\} < 20\epsilon.$$

For (iii), it suffices to extend $f - g$ on X_1 without increasing the norm [3, Lemma IIIa].

We use $S_r(x)$ to denote the closed sphere of radius r and center at x. If the center is 0, we simply use S_r instead.

LEMMA 3.2. *Let F be a closed subspace in a Banach space X, let $x_0 \in X \backslash F$, and let X_0 be the subspace generated by F and x_0. Suppose there exists a functional $f \in X_0^*$ such that $f^{-1}(0) = F$ and f exposes a point of the closed unit sphere of X_0; then F has property U_x for each $x \in X_0$.*

Proof. Suppose $\| f \| = 1$ and f exposes the closed unit sphere of X_0 at y_0. For each $x \in X_0 \backslash F$, we may assume that $f(x) = r > 0$ (otherwise, consider $-f$). We have

$$F \cap S_r(x) = \{x - ry_0\}.$$

This implies F has property U_x, i.e., $x - ry_0$ is the unique point in F satisfies

$$\| x - (x - ry_0) \| = \inf\{\| x - y \| : y \in F\}.$$

PROPOSITION 3.3. *Let X be a Banach space with property (P) and let F be a closed hyperplane in X. Then the set of Chebyshev subspaces in $\Theta(F)$ is a dense G_δ in $(\Theta(F), \rho)$.*

Proof. Note that the metric $\tilde{\rho}$ and ρ are equivalent in $\Theta(F)$. The conclusion follows from the definition of property (P), Lemma 3.1(b) and Lemma 3.2.

Suppose K_1, K_2 are bounded subsets in X, F is a closed subspace in X, and suppose $x \in X \backslash F$; we let

$$\operatorname{diam} K_1 = \sup\{\|x - y\| : x, y \in K_1\},$$
$$d(K_1, K_2) = \inf\{r : K_1 \subseteq K_2 + S_r,\ K_2 \subseteq K_1 + S_r\},$$

and for each $a > 0$, we define

$$C(a, x, F) = (F + (1 - a)x) \cap S_{\|x\|}.$$

It is clear that $\lim_{a \to 0} \operatorname{diam} C(a, x, F) = 0$ if and only if x is a strongly exposed point of $S_{\|x\|} \cap X_0$ where X_0 is the subspace generated by F and x; the corresponding strongly exposing functionals are the f in X_0^* with $f^{-1}(0) = F$. In such a case, by Lemma 3.1, F has property U_r.

LEMMA 3.4. *Let X be a Banach space and let F be a closed subspace in X. For each n, let*

$$U_n = \{(x, E) \in X \times \Theta(F) : \operatorname{diam} C(a, x, E) < 1/n \text{ for some } a > 0\}.$$

Then U_n is an open subset in $X \times \Theta(F)$ where $\Theta(F)$ has the metric topology defined by $\tilde{\rho}$.

Proof. Let $(x, E) \in U_n$. Without loss of generality, we assume $\|x\| = 1$. Let

$$\alpha = (1/n) - \operatorname{diam} C(a, x, E).$$

For $(x', E') \in X \times \Theta(F)$ with

$$\|x - x'\| \leqslant \alpha/16, \qquad \tilde{\rho}(E, E') < \alpha/16.$$

We have

$$C(a, x', E') \subseteq ((E + (1 - a)x) + S_{\alpha/8}) \cap S_{(1+\alpha/16)}$$
$$\subseteq ((E + (1 - a)x) \cap S_{(1+\alpha/4)}) + S_{\alpha/8}.$$

Also

$$d((E + (1 - a)x) \cap S_{(1+\alpha/4)}, C(a, x, E)) \leqslant d(S_{(1+\alpha/4)}, S_1)$$
$$= \alpha/4.$$

Hence

$$\operatorname{diam} C(a, x', E') \leqslant \operatorname{diam}((E + (1 - a)x)) \cap S_{(1+\alpha/4)}) + \operatorname{diam} S_{\alpha/8}$$
$$\leqslant \operatorname{diam} C(a, x, E) + \frac{\alpha}{2} + \frac{\alpha}{4}$$
$$= \frac{1}{n} - \alpha + \frac{3\alpha}{4}$$
$$< \frac{1}{n},$$

i.e., $(x', E') \in U_n$. This implies U_n is open.

THEOREM 3.5. *Let X be a separable Banach space with property (P). Let F be a closed subspace in X, then the set of almost Chebyshev subspaces in $\Theta(F)$ contains a dense G_δ in $(\Theta(F), \tilde{\rho})$.*

Proof. We first show that for each $x \in X$, there exists a dense subset in $\Theta(F)$ such that each member of the subset has property U_x. Indeed, for any $E \in \Theta(F)$ and for any $1 > \epsilon > 0$, consider the subspaces X_1 generated by E and x (assume that $x \notin E$, otherwise the result is trivial). Let $f \in X_1^*$ such that $\|f\| = 1$ and $f^{-1}(0) = E$. Note that X_1 also has property (P); there exists $g \in X_1^*$, a strongly exposing functional of the closed unit sphere of X_1 with $\|g\| = 1$ and $\|f - g\| < (20)^{-1}\epsilon$. Let $E' = g^{-1}(0)$, then $\tilde{\rho}(E, E') < \epsilon$ (Lemma 3.1) and E' has property U_x (Lemma 3.2).

For each x, let $\mathscr{D}_x$ denote the set of those members E' of $\Theta(F)$ which correspond to strongly exposing functionals of the unit balls of the subspaces generated by E and x, $E \in \Theta(F)$ (as above). Then $\mathscr{D}_x$ is dense in $\Theta(F)$. For each n, let

$$U_n = \{(x, E) \in X \times \Theta(F) : \operatorname{diam} C(a, x, E) < 1/n \text{ for some } a > 0\}.$$

By the remark preceding Lemma 3.4, we see that the x section of $\bigcap_{n=1}^{\infty} U_n$ equals $\mathscr{D}_x$. Hence Lemma 3.4 and the above imply that $\bigcap_{n=1}^{\infty} U_n$ is a dense G_δ; for each $(x, E) \in \bigcap_{n=1}^{\infty} U_n$, E has property U_x.

Note that X is a separable Banach space and $\Theta(F)$ is a complete metric space, Lemma 2.4 implies that there exists a dense G_δ subset $\mathscr{D}$ in $\Theta(F)$ with the property that for each $E \in \mathscr{D}$, there exists a dense G_δ subset D_E in X such that for $x \in D_E$, $(x, E) \in \bigcap_{n=1}^{\infty} U_n$. This means that each member in $\mathscr{D}$ is almost Chebyshev and we complete the proof.

By using the same proof as Lemma 3.4, we have

LEMMA 3.6. *Let X be a Banach space and let Θ be the family of closed subspaces in X. For each n, let*

$$V_n = \{(x, F) \in X \times \Theta : \operatorname{diam} C(a, x, F) < 1/n \text{ for some } a > 0\}$$

Then V_n is an open subset in $X \times \Theta$.

THEOREM 3.7. *Let X be a separable Banach space with property (P). Then the family of almost Chebyshev subspaces contains a dense G_δ in (Θ, ρ).*

Proof. For each $x \in X$, $E \in \Theta$, and $\epsilon > 0$, we can find a closed subspace E' which is B-isomophic to E, with property U_x and $\tilde{\rho}(E, E') < \epsilon$ (the first part of the proof of the last theorem). Note that $\rho(E, E') \leqslant \tilde{\rho}(E, E')$. This implies that the set of closed subspaces with property U_x is dense in Θ. Now consider the dense G_δ set $\bigcap_{n=1}^{\infty} V_n$ where $V_n = \{(x, E) \in X \times \Theta : \operatorname{diam} C(a, x, E) < 1/n$ for some $a > 0\}$. By exactly the same argument as last theorem, we conclude that the set of almost Chebyshev subspaces of X contains a dense G_δ in (Θ, ρ).

COROLLARY 3.8. *Let X be a separable Banach space satisfying either*

(a) *X has the RNP, or*

(b) *X has a locally uniformly convex norm.*

Then

(i) *for any closed subspace F, the family of almost Chebyshev subspaces which are B-isomorphic to F contains a dense G_δ in $(\Theta(F), \tilde{\rho})$,*

(ii) *the family of almost Chebyshev subspaces in X contains a dense G_δ in (Θ, ρ).*

4. Some Remarks

Theorems 3.5 and 3.7 partially generalized the results of Garkavi in considering the reflexive subspaces of separable Banach spaces and the weak* closed subspaces of separable conjugate Banach spaces. One of his examples (c_0) says that there exists a separable Banach space which does not have any nonreflexive almost Chebyshev subspace. Hence some restrictions on the subspaces or the Banach space are essential. The only place we use the separability is to prove Lemma 2.4. The lemma is not true without that condition. We are interested to know whether Theorems 3.5, 3.7 will still hold for Banach spaces with property (P) in general. In particular, are the theorems true for any reflexive Banach spaces?

It is proved by Stečkin [10] that in a uniformly convex Banach space, every closed subset is almost Chebyshev. (Note that the problem is trivial for closed convex sets in such spaces). Also, there are examples that there is a separable reflexive strictly convex space, and that the above result does not hold [13]. It is natural to ask: For what kind of spaces is it true that every closed subset is almost Chebyshev? Will it be true in a locally uniformly convex reflexive Banach space? Since Banach spaces with the RNP are characterized by the property that every bounded closed convex set is the closed convex hull of its strongly exposed points, it is also interesting to consider the best approximations for bounded closed sets in such spaces. Indeed, in [13], Edelstein prove that if K is a closed *convex* set in a Banach space with the RNP, then the set which admits best approximation in K is a weakly dense subset in X.

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